A note on the Painlevé analysis of a $(2+1)$ dimensional Camassa-Holm equation

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Abstract

We investigate the Painlevé analysis for a $(2+1)$ dimensional Camassa-Holm equation. Our results show that it admits only weak Painlevé expansions. This then confirms the limitations of the Painlevé test as a test for complete integrability when applied to non-semilinear partial differential equations.

Key words: Integrability, Painlevé analysis

1 Introduction

Recently there has been a great deal of interest in the study of the integrability properties of the Camassa-Holm (CH) equation,

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where subscripts denote partial derivatives and κ is a constant [1]. Equation (1) was first derived by Fokas and Fuchssteiner using the method of recursion operators [2]. However, it became more widely known when it was derived from physical considerations as a water wave equation by Camassa and Holm, by using asymptotic expansions directly in the Hamiltonian for Euler's equations governing inviscid incompressible flow in the shallow water regime [1]. They showed that equation (1) is completely integrable for arbitrary values of κ,

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and that for $\kappa = 0$ it admits a special kind of travelling wave solution, the peakon, of the form $ce^{-|x-ct|}$. Further, they showed that it possesses a Lax pair and bi-Hamiltonian structure. Subsequently, several works have been devoted to studying the underlying mathematical and physical properties of equation (1) [3,4,5,6,7,8,9,10,11,12,13].

Interestingly, several higher dimensional generalizations of the CH equation have also been proposed. For example, Kraenkel and Zenchuk constructed the following (2+1) dimensional integrable generalization of the CH equation [14],

$$
\begin{aligned}
&m_t - um_x - 2mu_x = 0,\\
&mu_y + u_x - u_{xx} - 2p_x = 0,\\
&m^2 p_y + mp_x + mp_{xx} - pm_x - m_x p_x = 0,
\end{aligned}
\tag{2}
$$

where m, u and p are functions of (x, y, t). In [14], equation (2) was derived from a system of $(2 + 1)$ dimensional shallow water equations by using a multiscale decomposition, and was also shown to have a Lax pair. The spectral problem for this equation was solved by Zenchuk [15]. Kraenkel et al. [16] studied the invariance properties of (2) through Lie symmetry analysis and explored its similarity reductions, giving Lax pairs for all the resulting $(1+1)$ dimensional partial differential equations (PDEs) and thus establishing their integrability. Recently, Johnson has derived another $(2 + 1)$ dimensional CH equation, which has a structure similar to that of the Kadomtsev-Petviashvili equation, again within the context of equations governing water waves [17]. The integrability of this equation is discussed in [18].

Even though many interesting mathematical properties have been studied for the (2+1) dimensional CH equation (2), an important question remains unanswered, that is, whether or not it passes the Painlevé (P-) test [19,20]. Indeed, certain difficulties arise when applying the P-test to equation (2). These difficulties are similar to those that arise for the class of equations studied in [4], and which also arise for the special case of the CH equation itself. However, in [4] it was shown how these difficulties can be overcome by including an extra lower order term in the Painlevé expansion; in particular, it was shown that the CH equation admits only weak Painlevé expansions. We find that a similar modification of the leading order analysis allows us to apply the P-test to equation (2). We find that this equation admits only weak Painlevé expansions.

2 Painlevé analysis

In this section we apply the P-test [19,20] to the $(2+1)$ dimensional CH equation (2). The P-test consists essentially of three steps: (i) determination of leading-order behaviours, (ii) identifying the resonances, and (iii) checking the corresponding compatibility conditions (in order to avoid logarithmic branching).

2.1 (i) Leading order analysis

As a first step we consider seeking local Laurent expansions about a noncharacteristic movable singular manifold $\varphi = 0$. We find that the leading order exponent for u cannot be negative, and so assume, following [4], that the leading orders of the solutions of equation (2) are

$$u \sim c\psi_t + u_0\phi^\alpha, \qquad m \sim m_0\phi^\beta, \qquad p \sim p_0\phi^\gamma, \tag{3}$$

where c is a constant to be determined. We note that here we are using Kruskal's ansatz [21], that is, we are taking $\phi = x + \psi(y,t)$ and all coefficients in our Laurent expansions (for example, u_0, m_0 and p_0 above) to be functions of (y,t) only.

Balancing the most dominant terms, we find that $c = 1$, and are led to the following two cases:

Case 1 $\alpha = \frac{1}{2}$, $\beta = -1$, $\gamma = -\frac{1}{2}$ with $m_0 = \frac{1}{2\psi_y}$, $p_0 = -\frac{u_0}{2}$, u_0 arbitrary.

Case 2 $\alpha = \frac{1}{2}$, $\beta = -1$, $\gamma = \frac{1}{2}$ with $m_0 = -\frac{1}{2\psi_y}$, u_0 and p_0 arbitrary.

2.2 (ii) Resonances

The next step in the P-test is to find the resonances, that is, the powers at which arbitrary functions appear in the series. As we have encountered two branches in the leading order analysis, we present the calculations for each case separately.

Case 1

Substituting the expressions

$$
\begin{aligned}
&u=\psi_t+u_0\phi^{\frac{1}{2}}+u_r\phi^{r+\frac{1}{2}},\\
&m=m_0\phi^{-1}+m_r\phi^{r-1},\\
&p=p_0\phi^{-\frac{1}{2}}+p_r\phi^{r-\frac{1}{2}},
\end{aligned}
\tag{4}
$$

into the dominant terms tells us that the recursion relation for the coefficients of our Laurent expansions is of the form

$$
\begin{pmatrix}
\frac{r}{\psi_y} & ru_0 & 0\\
-\frac{1}{2}(2r+1)(r-1) & \frac{1}{2}u_0\psi_y & 1-2r\\
0 & -\frac{1}{8}(2r-1)u_0 & \frac{r(2r-1)}{4\psi_y}
\end{pmatrix}
\begin{pmatrix} u_r\\ m_r\\ p_r\end{pmatrix}
=
\begin{pmatrix} f_r\\ g_r\\ h_r\end{pmatrix},
\tag{5}
$$

where as usual f_r, g_r and h_r are functions of ψ, previous coefficients, and derivatives thereof. We thus obtain the resonances as $r=-1,0,\frac{1}{2},\frac{1}{2},1$. The resonance $r=-1$ is associated with the arbitrariness of the function ψ; $r=0$ corresponds to the function u_0 being arbitrary. The double resonance $r=\frac{1}{2}$ indicates that we must modify our Laurent series in order to include half-integer powers of ϕ, thus obtaining a Puiseux series, and that two of the coefficients $(u_{\frac{1}{2}},m_{\frac{1}{2}},p_{\frac{1}{2}})$ therein will be arbitrary. Finally the resonance $r=1$ indicates that one of the coefficients (u_1,m_1,p_1) will be arbitrary.

Case 2

In a similar way, substituting the expressions

$$
\begin{aligned}
&u=\psi_t+u_0\phi^{\frac{1}{2}}+u_r\phi^{r+\frac{1}{2}},\\
&m=m_0\phi^{-1}+m_r\phi^{r-1},\\
&p=p_0\phi^{\frac{1}{2}}+p_r\phi^{r+\frac{1}{2}},
\end{aligned}
\tag{6}
$$

into the dominant terms of (2) tells us that the form of the recursion relation is

$$
\begin{pmatrix}
-\frac{r}{\psi_y} & ru_0 & 0\\
-r(r+\frac{1}{2}) & \frac{1}{2}u_0\psi_y & 0\\
0 & -\frac{p_0}{2}(r+\frac{1}{2}) & -\frac{r}{2\psi_y}(r+\frac{1}{2})
\end{pmatrix}
\begin{pmatrix} u_r\\ m_r\\ p_r\end{pmatrix}
=
\begin{pmatrix} \tilde{f}_r\\ \tilde{g}_r\\ \tilde{h}_r\end{pmatrix},
\tag{7}
$$

where the functions $\tilde{f}_r$, $\tilde{g}_r$ and $\tilde{h}_r$ are functions of ψ, previous coefficients, and derivatives thereof. We thus obtain the resonances $r=-1,-\frac{1}{2},0,0,\frac{1}{2}$. The double resonance $r=0$ corresponds to the arbitrariness of the functions u_0 and p_0. The resonance $r=\frac{1}{2}$ indicates once again that we need to modify our Laurent series in order to include half-integer powers of ϕ, and that one of

the coefficients $(u_{\frac{1}{2}}, m_{\frac{1}{2}}, p_{\frac{1}{2}})$ in the resulting series will be arbitrary. We might expect that the resonance $r = -\frac{1}{2}$ means that we need to use the perturbative Painlevé test. However, we will see that this resonance, on this occasion, can be accommodated much more easily.

We note that, since we have non-integer resonances (in fact, in each of the above cases), equation (2) does not pass the Painlevé test. It remains to check whether or not it admits weak Painlevé — or Puiseux — expansions (see [20], also [4,22]).

2.3 (iii) Checking compatibility conditions

We now check the compatibility conditions, using the following expansions.

Case 1

We substitute

$$u = \psi_t + \phi^{\frac{1}{2}} \sum_{j=0}^{\infty} u_{\frac{j}{2}} \phi^{\frac{j}{2}}, \quad m = \phi^{-1} \sum_{j=0}^{\infty} m_{\frac{j}{2}} \phi^{\frac{j}{2}}, \quad p = \phi^{-\frac{1}{2}} \sum_{j=0}^{\infty} p_{\frac{j}{2}} \phi^{\frac{j}{2}}, \tag{8}$$

with $\phi = x + \psi(y,t)$ and coefficients $u_i = u_i(y,t)$, $m_j = m_j(y,t)$ and $p_k = p_k(y,t)$, and with $m_0 = \frac{1}{2\psi_y}$, $p_0 = -\frac{u_0}{2}$ and u_0 arbitrary, into equation (2).

At the point of determining the coefficients $(u_{\frac{1}{2}}, m_{\frac{1}{2}}, p_{\frac{1}{2}})$, we find that $p_{\frac{1}{2}}$ is arbitrary, and that one of $u_{\frac{1}{2}}$ and $m_{\frac{1}{2}}$ is also arbitrary, with both compatibility conditions satisfied. We choose to leave $m_{\frac{1}{2}}$ arbitrary, solving for $u_{\frac{1}{2}}$ as

$$u_{\frac{1}{2}} = -\frac{\psi_{yt}}{\psi_y} - u_0 m_{\frac{1}{2}} \psi_y. \tag{9}$$

At the point of determining the coefficients (u_1, m_1, p_1), we find that any one of these three can be left arbitrary with satisfied compatibility condition. Thus for example we may choose m_1 to be arbitrary, and solve for u_1 and p_1. All subsequent coefficients are then determined in terms of the five arbitrary functions $(\psi, u_0, m_{\frac{1}{2}}, p_{\frac{1}{2}}, m_1)$. Since all compatibility conditions are satisfied, we see that in this case, equation (2) admits a weak Painlevé expansion.

Case 2

In this case, instead of the series that would usually be chosen to check com-

patibility conditions, we substitute the series

$$u = \psi_t + \phi^{\frac{1}{2}} \sum_{j=0}^{\infty} u_{\frac{j}{2}} \phi^{\frac{j}{2}}, \quad m = \phi^{-1} \sum_{j=0}^{\infty} m_{\frac{j}{2}} \phi^{\frac{j}{2}}, \quad p = p_{-\frac{1}{2}} + \phi^{\frac{1}{2}} \sum_{j=0}^{\infty} p_{\frac{j}{2}} \phi^{\frac{j}{2}}, \tag{10}$$

with $m_0 = -\frac{1}{2\psi_y}$, and u_0 and p_0 arbitrary, into equation (2). The difference with the usual series that would be taken lies in the inclusion of the extra term $p_{-\frac{1}{2}}$. It is the inclusion of this extra term that allows us to accommodate the resonance $r = -\frac{1}{2}$, since we find that $p_{-\frac{1}{2}}$ is left arbitrary.

At the point of determining the coefficients $(u_{\frac{1}{2}}, m_{\frac{1}{2}}, p_{\frac{1}{2}})$, we find that any one of these three can be left arbitrary with satisfied compatibility condition. Thus for example we may choose $m_{\frac{1}{2}}$ to be arbitrary, and solve for $u_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$. All subsequent coefficients are then determined in terms of the five arbitrary functions $(\psi, p_{-\frac{1}{2}}, u_0, p_0, m_{\frac{1}{2}})$. Since all compatibility conditions are satisfied, we see that also in this case, equation (2) admits a weak Painlevé expansion.

3 Conclusions

In this letter we have carried out a detailed investigation of the Painlevé analysis of the completely integrable $(2+1)$ dimensional generalization of the CH equation, equation (2). We have used a modified version of the leading order analysis, as adopted in [4]. In this way, we have shown that the equation under study admits only weak Painlevé expansions. This example therefore confirms the limitation of the Painlevé test as a test for complete integrability when applied to non-semilinear PDEs.

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References

[1] R. Camassa and D. D. Holm, *Phys. Rev. Letts.* **71**, 1661, (1993);
R. Camassa, D. D. Holm and J. M. Hyman, *Adv. Appl. Mech.* **31**, 1, (1994).

[2] B. Fuchssteiner and A. S. Fokas, *Physica* **D4**, 47, (1981).

[3] F. Cooper and H. Shepard, *Phys. Lett. A* **194**, 246, (1994).

[4] C. Gilson and A. Pickering, *J. Phys. A: Math. Gen.* **28**, 2871, (1995).

[5] B. Fuchssteiner, *Physica D* **95**, 229, (1996).

[6] M. Senthil Velan and M. Lakshmanan, *Int. J. Nonlinear Mech.* **31**, 339, (1996).

[7] J. Schiff, *Physica D* **121**, 24, (1998).

[8] M. Fisher and J. Schiff, *Phys. Lett. A* **259**, 371, (1999).

[9] A. Constantin, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci* **457**, 953, (2001).

[10] Z. Liu and J. Qian, *Int. J. Bifurcation and Chaos* **11**, 781, (2001).

[11] A. Pickering, *Theor. Math. Phys.* **135**, 638, (2003).

[12] R. Camassa, *Discrete and Continuous Dynamical Systems* **3**, 115, (2003).

[13] H. R. Dullin, G. A. Gottwald and D. D. Holm, *Fluid Dynamics Research* **33**, 73, (2003).

[14] R. A. Kraenkel and A. I. Zenchuk, *Phys. Lett. A* **260**, 218, (1999).

[15] A. I. Zenchuk, *Physica D* **152-153**, 178, (2001).

[16] R. A. Kraenkel, M. Senthilvelan and A. I. Zenchuk, *Phys. Lett. A* **273**, 183, (2000).

[17] R. S. Johnson, *J. Fluid Mech.* **455**, 63, (2002).

[18] P. R. Gordoa, A. Pickering and M. Senthilvelan, *Z. Naturforsch. A* **59**, 640, (2004).

[19] J. Weiss, M. Tabor and G. Carnevale, *J. Math. Phys.* **24**, 522, (1983).

[20] J. Weiss, *J. Math. Phys.* **24**, 1405, (1983).

[21] M. Jimbo, M. D. Kruskal and T. Miwa, *Phys. Lett. A* **92**, 59, (1982).

[22] A. Pickering, *Phys. Lett. A* **221**, 174, (1996).